Non-linear gravity wave interactions

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In earlier papers Phillips (1960) and Longuet-Higgins (1962) have investigated phase velocity effects and possible resonances associated with the interactions of gravity waves. In this note the problem is discussed from a different viewpoint which demonstrates more clearly the energy-sharing mechanism involved. Equations governing the time dependence of the resonant modes are obtained, rather than the initial growth rate as has been found previously.

1. Introduction

Consider the classical problem of wave propagation on deep water, namely the solution of

$$\Delta\phi = 0, \tag{1.1}$$

subject to the kinematic and pressure conditions

$$\eta_t + \eta_x\phi_x + \eta_z\phi_z - \phi_y = 0, \quad \text{at} \quad y = \eta(x,z,t), \tag{1.2}$$

$$\phi_t + g\eta + \tfrac{1}{2}|\nabla\phi|^2 = 0, \quad \text{at} \quad y = \eta(x,z,t), \tag{1.3}$$

$$|\nabla\phi| \to 0, \quad \text{as} \quad y \to -\infty. \tag{1.4}$$

Here $y = 0$ is taken as the undisturbed and $y = \eta(x,z,t)$ as the disturbed free surface, g denotes gravity and $\mathbf{q} = \nabla\phi$.

The standard procedure of solving this problem is first to replace the boundary conditions (1.2) and (1.3), which are applied at the unknown surface $y = \eta$, by equivalent ones at the known undisturbed surface $y = 0$. Quantities up to and including the third-order terms only will be retained in the subsequent analysis. It is found that equations (1.2) and (1.3) are to be replaced by

$$\eta_t - \phi_y + (\eta\phi_x)_x + (\eta\phi_z)_z + \tfrac{1}{2}(\eta^2\phi_{xy})_x + \tfrac{1}{2}(\eta^2\phi_{zy})_z = 0, \quad \text{at} \quad y = 0, \tag{1.5}$$

$$g\eta + \phi_t + \eta\phi_{yt} + \tfrac{1}{2}|\nabla\phi|^2 + \tfrac{1}{2}\eta^2\phi_{yyt} + \tfrac{1}{2}\eta|\nabla\phi|^2_y = 0, \quad \text{at} \quad y = 0. \tag{1.6}$$

Elimination of η then yields a boundary condition on ϕ alone, namely,

$$\begin{aligned}\phi_{tt} + g\phi_y = {} & \{-\tfrac{1}{2}|\nabla\phi|^2 + g^{-1}\phi_t\phi_{yt} + \tfrac{1}{2}g^{-1}(\phi_t|\nabla\phi|^2)_y - g^{-2}\phi_t\phi_{yt}^2 - \tfrac{1}{2}g^{-2}\phi_{yyt}\phi_t^2\}_t \\ & + \{-\phi_x\phi_t - \tfrac{1}{2}\phi_x|\nabla\phi|^2 + g^{-1}\phi_x\phi_t\phi_{yt} + \tfrac{1}{2}g^{-1}\phi_t^2\phi_{xy}\}_x \\ & + \{-\phi_z\phi_t - \tfrac{1}{2}\phi_z|\nabla\phi|^2 + g^{-1}\phi_z\phi_t\phi_{yt} + \tfrac{1}{2}g^{-1}\phi_t^2\phi_{zy}\}_z, \quad \text{at} \quad y = 0.\end{aligned} \tag{1.7}$$

To discuss the interactions of a given set of primary waves we write

$$\begin{aligned}\phi = {} & A_{0,0}(t) + \sum_l (A_l(t)\, e^{i\mathbf{k}_l\cdot\mathbf{r}} + A_{-l}(t)\, e^{-i\mathbf{k}_l\cdot\mathbf{r}})\, e^{|\mathbf{k}_l|y} \\ & + \sum_{l,m} (A_{l,m}(t)\, e^{i(\mathbf{k}_l+\mathbf{k}_m)\cdot\mathbf{r}} + A_{-l,-m}(t)\, e^{-i(\mathbf{k}_l+\mathbf{k}_m)\cdot\mathbf{r}})\, e^{|\mathbf{k}_l+\mathbf{k}_m|y} \\ & + \sum_{l,m,n} (A_{l,m,n}(t)\, e^{i(\mathbf{k}_l+\mathbf{k}_m+\mathbf{k}_n)\cdot\mathbf{r}} + A_{-l,-m,-n}(t)\, e^{-i(\mathbf{k}_l+\mathbf{k}_m+\mathbf{k}_n)\cdot\mathbf{r}})\, e^{|\mathbf{k}_l+\mathbf{k}_m+\mathbf{k}_n|y},\end{aligned} \tag{1.8}$$

where $\mathbf{k}_l$ and $\mathbf{r}$ denote the wave-number and radius vectors in the (x, z)-plane. The functions $A_{-l}, A_{-l,-m,\ldots}$ are the complex conjugates of $A_l, A_{l,m,\ldots}$ respectively, and the number of subscripts is the order of the wave concerned.

The form assumed for ϕ in equation (1.8) satisfies equations (1.1) and (1.4) and it remains to satisfy the boundary condition (1.7).

If the non-linear terms in equation (1.7) are neglected we have the familiar equations,

$$\frac{d^2A_l}{dt^2}+\omega_l^2A_l = 0 \quad (l = 1, 2, \ldots), \tag{1.9}$$

where $\omega_l^2 = g\,|\mathbf{k}_l|$, corresponding to the classical linear theory. Here each mode propagates with velocity $\pm c_l$, where $c_l = (g/|\mathbf{k}_l|)^{\frac{1}{2}}$, independent of the other modes. Equation (1.9) corresponds to a system of uncoupled linear oscillators, the general solution being

$$A_l = a_l e^{-i\omega_l t}+b_l e^{i\omega_l t}, \tag{1.10}$$

where a_l and b_l are arbitrary complex constants describing the amplitudes of the two antiparallel waves. Later as we shall see the inclusion of the non-linear terms of equation (1.7) necessitates taking a_l and b_l as slowly varying functions of time.

Using equations (1.7) and (1.8), the differential equations for the second-order motion are found to be

$$\frac{d^2A_{0,0}}{dt^2} = 2\frac{d}{dt}\sum_m\left\{\frac{|\mathbf{k}_m|^2}{\omega_m^2}\frac{dA_m}{dt}\frac{dA_{-m}}{dt}-|\mathbf{k}_m|^2A_mA_{-m}\right\}, \tag{1.11}$$

$$\frac{d^2A_{l,m}}{dt^2}+g\,|\mathbf{k}_l+\mathbf{k}_m|\,A_{l,m} = 2\{\mathbf{k}_l.\mathbf{k}_m-|\mathbf{k}_l|\,|\mathbf{k}_m|\}\frac{d}{dt}(A_lA_m). \tag{1.12}$$

From equation (1.12) we have,

$$\begin{aligned}A_{l,m} = {}& \frac{2i(\omega_l+\omega_m)\,(|\mathbf{k}_l|\,|\mathbf{k}_m|-\mathbf{k}_l.\mathbf{k}_m)}{g\,|\mathbf{k}_l+\mathbf{k}_m|-(\omega_l+\omega_m)^2}(a_la_me^{-i(\omega_l+\omega_m)t}-b_lb_me^{i(\omega_l+\omega_m)t})\\ &+\frac{2i(\omega_l-\omega_m)\,(|\mathbf{k}_l|\,|\mathbf{k}_m|-\mathbf{k}_l.\mathbf{k}_m)}{g\,|\mathbf{k}_l+\mathbf{k}_m|-(\omega_l-\omega_m)^2}(a_lb_me^{-i(\omega_l-\omega_m)t}-a_mb_le^{i(\omega_l-\omega_m)t}),\end{aligned} \tag{1.13}$$

where only the oscillations forced by the primary modes have been included.

To find the non-linear oscillator equations governing the primary modes we first evaluate the various coefficients on the right-hand side of equation (1.7), using the assumed form for ϕ.

For $e^{i\mathbf{k}_l.\mathbf{r}}$ we have

$$\begin{aligned}&\frac{d}{dt}\left\{\frac{|\mathbf{k}_l|}{g}\frac{dA_l}{dt}\frac{dA_{0,0}}{dt}-\frac{9}{2g^2}|\mathbf{k}_l|^2\left(\frac{dA_l}{dt}\right)^2\frac{dA_{-l}}{dt}+\frac{6\,|\mathbf{k}_l|^3}{g}A_lA_{-l}\frac{dA_l}{dt}\right\}\\ &\quad+|\mathbf{k}_l|^2A_l\frac{dA_{0,0}}{dt}+2\,|\mathbf{k}_l|^4A_l^2A_{-l}-\frac{3\,|\mathbf{k}_l|^3}{g}A_l\frac{dA_l}{dt}\frac{dA_{-l}}{dt}+\frac{3}{2}\frac{|\mathbf{k}_l|^3}{g}A_{-l}\left(\frac{dA_l}{dt}\right)^2\\ &\quad+\sum_{m(\neq l)}\left[\frac{d}{dt}\left\{(\mathbf{k}_m.(\mathbf{k}_l-\mathbf{k}_m)-|\mathbf{k}_m|\,|\mathbf{k}_l-\mathbf{k}_m|)\,A_mA_{l,m}\right.\right.\\ &\quad-(\mathbf{k}_m.(\mathbf{k}_l+\mathbf{k}_m)+|\mathbf{k}_m|\,|\mathbf{k}_l+\mathbf{k}_m|)\,A_{-m}A_{l,m}\\ &\quad+\frac{1}{g}(|\mathbf{k}_m|+|\mathbf{k}_l-\mathbf{k}_m|)\frac{dA_m}{dt}\frac{dA_{l,-m}}{dt}+\frac{1}{g}(|\mathbf{k}_m|+|\mathbf{k}_l+\mathbf{k}_m|)\frac{dA_{-m}}{dt}\frac{dA_{l,m}}{dt}\end{aligned}$$

$$-\frac{1}{g^2}(|\mathbf{k}_l|+2|\mathbf{k}_m|)\frac{dA_l}{dt}\frac{dA_m}{dt}\frac{dA_{-m}}{dt}+\frac{2|\mathbf{k}_m|^2}{g}(|\mathbf{k}_l|+2|\mathbf{k}_m|)A_m A_{-m}\frac{dA_l}{dt}$$

$$+\frac{1}{g}(|\mathbf{k}_l||\mathbf{k}_m|+\mathbf{k}_l.\mathbf{k}_m)(|\mathbf{k}_l|+2|\mathbf{k}_m|)A_l A_{-m}\frac{dA_m}{dt}$$

$$+\frac{1}{g}(|\mathbf{k}_l||\mathbf{k}_m|-\mathbf{k}_l.\mathbf{k}_m)(|\mathbf{k}_l|+2|\mathbf{k}_m|)A_l A_m\frac{dA_{-m}}{dt}\Big\}$$

$$+(\mathbf{k}_l.\mathbf{k}_m)A_m\frac{dA_{l,-m}}{dt}-(\mathbf{k}_l.\mathbf{k}_m)A_{-m}\frac{dA_{l,m}}{dt}+\mathbf{k}_l.(\mathbf{k}_l-\mathbf{k}_m)\frac{dA_m}{dt}A,-,$$

$$+\mathbf{k}_l.(\mathbf{k}_l+\mathbf{k}_m)\frac{dA_{-m}}{dt}A_{l,m}-\frac{|\mathbf{k}_l|^2}{g}(|\mathbf{k}_l|+2|\mathbf{k}_m|)A_l\frac{dA_m}{dt}\frac{dA_{-m}}{dt}$$

$$+2(|\mathbf{k}_l|^2|\mathbf{k}_m|^2+(\mathbf{k}_l.\mathbf{k}_m)^2A_lA_mA_{-m}-\frac{(\mathbf{k}_l.\mathbf{k}_m)}{g}(|\mathbf{k}_l|+2|\mathbf{k}_m|)A_m\frac{dA_l}{dt}\frac{dA_{-m}}{dt}$$

$$+\frac{(\mathbf{k}_l.\mathbf{k}_m)}{g}(|\mathbf{k}_l|+2|\mathbf{k}_m|)A_{-m}\frac{dA_l}{dt}\frac{dA_m}{dt}\Big]. \tag{1.14}$$

The coefficient of $e^{i(\mathbf{k}_l+\mathbf{k}_m+\mathbf{k}_n).\mathbf{r}}$ is

$$\frac{d}{dt}\Big\{(\mathbf{k}_l.(\mathbf{k}_m+\mathbf{k}_n)-|\mathbf{k}_l||\mathbf{k}_m+\mathbf{k}_n|)A_lA_{m,n}+(\mathbf{k}_m.(\mathbf{k}_n+\mathbf{k}_l)-|\mathbf{k}_m||\mathbf{k}_n+\mathbf{k}_l|)A_mA_{n,l}$$

$$+(\mathbf{k}_n.(\mathbf{k}_l+\mathbf{k}_m)-|\mathbf{k}_n||\mathbf{k}_l+\mathbf{k}_m|)A_nA_{l,m}+\frac{1}{g}(|\mathbf{k}_l|+|\mathbf{k}_m+\mathbf{k}_n|)\frac{dA_l}{dt}\frac{dA_{m,n}}{dt}$$

$$+\frac{1}{g}(|\mathbf{k}_m|+|\mathbf{k}_n+\mathbf{k}_l|)\frac{dA_m}{dt}\frac{dA_{n,l}}{dt}+\frac{1}{g}(|\mathbf{k}_n|+|\mathbf{k}_l+\mathbf{k}_m|)\frac{dA_n}{dt}\frac{dA_{l,m}}{dt}$$

$$-\frac{1}{g^2}(|\mathbf{k}_l|+|\mathbf{k}_m|+|\mathbf{k}_n|)^2A_lA_mA_n+\frac{1}{g}(|\mathbf{k}_l|+|\mathbf{k}_m|+|\mathbf{k}_n|)$$

$$\times(|\mathbf{k}_m||\mathbf{k}_n|-\mathbf{k}_m.\mathbf{k}_n)\frac{dA_l}{dt}A_mA_n$$

$$+\frac{1}{g}(|\mathbf{k}_l|+|\mathbf{k}_m|+|\mathbf{k}_n|)(|\mathbf{k}_n||\mathbf{k}_l|-\mathbf{k}_n.\mathbf{k}_l)\frac{dA_m}{dt}A_nA_l$$

$$+\frac{1}{g}(|\mathbf{k}_l|+|\mathbf{k}_m|+|\mathbf{k}_n|)(|\mathbf{k}_l||\mathbf{k}_m|-\mathbf{k}_l.\mathbf{k}_m)\frac{dA_n}{dt}A_lA_m\Big\}$$

$$+(\mathbf{k}_l+\mathbf{k}_m+\mathbf{k}_n).\Big\{\mathbf{k}_lA_l\frac{dA_{m,n}}{dt}+\mathbf{k}_mA_m\frac{dA_{n,l}}{dt}+\mathbf{k}_nA_n\frac{dA_{l,m}}{dt}\Big\}$$

$$+(\mathbf{k}_l+\mathbf{k}_m+\mathbf{k}_n).\Big\{(\mathbf{k}_m+\mathbf{k}_n)A_{m,n}\frac{dA_l}{dt}+(\mathbf{k}_n+\mathbf{k}_l)A_{n,l}\frac{dA_m}{dt}+(\mathbf{k}_l+\mathbf{k}_m)A_{l,m}\frac{dA_n}{dt}\Big\}$$

$$-\frac{(|\mathbf{k}_l|+|\mathbf{k}_m|+|\mathbf{k}_n|)}{9}(\mathbf{k}_l+\mathbf{k}_m+\mathbf{k}_n)$$

$$\Big\{\mathbf{k}_iA_l\frac{dA_m}{dt}\frac{dA_n}{dt}+\mathbf{k}_mA_m\frac{dA_n}{dt}\frac{dA_l}{dt}+\mathbf{k}_nA_n\frac{dA_l}{dt}\frac{dA_m}{dt}\Big\}$$

$$+(\mathbf{k}_l+\mathbf{k}_m+\mathbf{k}_n).\{(|\mathbf{k}_m||\mathbf{k}_n|-\mathbf{k}_m.\mathbf{k}_n)\mathbf{k}_l+(|\mathbf{k}_n||\mathbf{k}_l|-\mathbf{k}_n.\mathbf{k}_l)\mathbf{k}_m$$

$$+(|\mathbf{k}_i||\mathbf{k}_m|-\mathbf{k}_l.\mathbf{k}_m)\mathbf{k}_n\}A_lA_mA_n. \tag{1.15}$$

The coefficient of $e^{i(2\mathbf{k}_l+\mathbf{k}_m)\cdot\mathbf{r}}$ is

$$\frac{d}{dt}\left\{(\mathbf{k}_l\cdot(\mathbf{k}_l+\mathbf{k}_m)-|\mathbf{k}_l|\,|\mathbf{k}_l+\mathbf{k}_m|)\,A_l A_{l,m}+\frac{1}{g}(|\mathbf{k}_l|+|\mathbf{k}_l+\mathbf{k}_m|)\frac{dA_l}{dt}\frac{dA_{l,m}}{dt}\right.$$
$$\left.-\frac{1}{2g^2}(2|\mathbf{k}_l|+|\mathbf{k}_m|)^2\left(\frac{dA_l}{dt}\right)^2\frac{dA_m}{dt}+\frac{1}{g}(2|\mathbf{k}_l|+|\mathbf{k}_m|)(|\mathbf{k}_l|\,|\mathbf{k}_m|-\mathbf{k}_l\cdot\mathbf{k}_m)\,A_l A_m\frac{dA_l}{dt}\right\}$$
$$+\mathbf{k}_l\cdot(2\mathbf{k}_l+\mathbf{k}_m)\,A_l\frac{dA_{l,m}}{dt}+(\mathbf{k}_l+\mathbf{k}_m)\cdot(2\mathbf{k}_l+\mathbf{k}_m)\,A_{l,m}\frac{dA_l}{dt}$$
$$-\frac{\mathbf{k}_l\cdot(2\mathbf{k}_l+\mathbf{k}_m)(2|\mathbf{k}_l|+|\mathbf{k}_m|)}{g}A_l\frac{dA_l}{dt}\frac{dA_m}{dt}-\frac{\mathbf{k}_m\cdot(2\mathbf{k}_l+\mathbf{k}_m)(2|\mathbf{k}_l|+|\mathbf{k}_m|)}{2g}A_m\left(\frac{dA_l}{dt}\right)^2$$
$$+\mathbf{k}_l\cdot(2\mathbf{k}_l+\mathbf{k}_m)(|\mathbf{k}_l|\,|\mathbf{k}_m|-\mathbf{k}_l\cdot\mathbf{k}_m)\,A_l^2A_m. \tag{1.16}$$

2. Case of no apparent resonances

In this section we consider the case in which

$$\omega_l\pm\omega_m\pm\omega_n\pm\omega_p \quad \text{and} \quad \mathbf{k}_l\pm\mathbf{k}_m\pm\mathbf{k}_n\pm\mathbf{k}_p$$

(with the same sign sequence in each) are not both zero or close to zero (in the sense that they differ from zero by terms of the second order). This means that the third-order non-linear interactions do not generate a natural mode of the system.

The oscillator equations governing the time dependence of the primary modes are

$$\frac{d^2A_l}{dt^2}+\omega_l^2A_l=F_l \quad (l=1,2,\ldots), \tag{2.1}$$

where F_l is a cubic function of the A_m and their derivatives; namely the expression (1.14). Such a system can be solved by the asymptotic methods of non-linear mechanics; e.g. Bogoliuboff & Mitropolski (1958). We write

$$A_l=a_le^{-i\omega_lt}+b_le^{i\omega_lt}, \tag{2.2}$$

where a_l and b_l are now considered to be slowly varying functions of time. The small parameter is the maximum slope of the disturbed surface. One then takes

$$\frac{dA_l}{dt}=-i\omega_la_le^{-i\omega_lt}+i\omega_lb_le^{i\omega_lt}, \tag{2.3}$$

with

$$\frac{da_l}{dt}e^{-i\omega_lt}+\frac{db_l}{dt}e^{i\omega_lt}=0. \tag{2.4}$$

Using equation (2.1) we have

$$-i\omega_l\frac{da_l}{dt}e^{-i\omega_lt}+i\omega_l\frac{db_l}{dt}e^{i\omega_lt}=F_l, \tag{2.5}$$

and so,

$$\omega_l\frac{da_l}{dt}=\frac{iF_l}{2}e^{i\omega_lt}, \tag{2.6}$$

and

$$\omega_l\frac{db_l}{dt}=-\frac{iF_l}{2}e^{-i\omega_lt}. \tag{2.7}$$

Equation (1.13) and its counterpart derivable from equation (1.11) are still valid for the second-order motion but the a_m and b_m are now considered as time dependent.

From equations (2.6) and (2.7) the theory of the first approximations gives

$$\omega_l \frac{da_l}{dt} = ia_l \sum_n (\alpha_{lm} a_m a_m^* + \beta_{lm} b_m b_m^*) \quad (l=1,2,\ldots.), \tag{2.8}$$

$$\omega_l \frac{db_l}{dt} = -ib_l \sum_m (\gamma_{lm} a_m a_m^* + \delta_{lm} b_m b_m^*) \quad (l=1,2,\ldots), \tag{2.9}$$

where each of the α_{lm}, β_{lm}, γ_{lm}, δ_{lm} are real functions of the wave-numbers and frequencies concerned. In fact a little algebra shows that

$$\alpha_{lm} = \beta_{lm} = \gamma_{lm} = \delta_{lm} = -2|\mathbf{k}_l|^4, \quad \text{if} \quad l=m, \tag{2.10}$$

$$\begin{aligned}\alpha_{lm} = \beta_{lm} = \gamma_{lm} = \delta_{lm} = {} & |\mathbf{k}_l|^2|\mathbf{k}_m|^2 + (\mathbf{k}_l.\mathbf{k}_m)^2 - 2\frac{\omega_l\omega_m}{g}(\mathbf{k}_l.\mathbf{k}_m)(|\mathbf{k}_l| + 2|\mathbf{k}_m|) \\ & + \frac{(\omega_l+\omega_m)(|\mathbf{k}_l|\,|\mathbf{k}_m| - \mathbf{k}_l.\mathbf{k}_m)}{(\omega_l+\omega_m)^2 - g|\mathbf{k}_l+\mathbf{k}_m|} \\ & \qquad \times\{2\omega_l\mathbf{k}_l.\mathbf{k}_m - \omega_m(|\mathbf{k}_l|\,|\mathbf{k}_m| + |\mathbf{k}_l|\,|\mathbf{k}_l+\mathbf{k}_m| - |\mathbf{k}_l|^2 - 2\mathbf{k}_l.\mathbf{k}_m)\} \\ & + \frac{(\omega_l-\omega_m)(|\mathbf{k}_l|\,|\mathbf{k}_m| + \mathbf{k}_l.\mathbf{k}_m)}{(\omega_l-\omega_m)^2 - g|\mathbf{k}_l-\mathbf{k}_m|} \\ & \qquad \times\{-2\omega_l\mathbf{k}_l.\mathbf{k}_m + \omega_m(|\mathbf{k}_l|\,|\mathbf{k}_m| + |\mathbf{k}_l|\,|\mathbf{k}_l-\mathbf{k}_m| - |\mathbf{k}_l|^2 + 2\mathbf{k}_l.\mathbf{k}_m)\},\end{aligned} \tag{2.11}$$

if $l \neq m$.

The equations (2.8) and (2.9) can be solved explicitly to give

$$a_l = a_l(0)\,e^{-i\omega_l\lambda_l t}, \tag{2.12}$$

$$b_l = b_l(0)\,e^{i\omega_l\lambda_l t}, \tag{2.13}$$

where
$$\omega_l^2\lambda_l = -\sum_m \alpha_{lm}(a_m(0)\,a_m^*(0) + b_m(0)\,b_m^*(0)). \tag{2.14}$$

The expressions λ_l give the dependence of the frequencies on the amplitudes, or equivalently the $\lambda_l/|\mathbf{k}_l|$ give the modifications to the wave speeds $\pm c_l$ of the linear modes. If there is only one mode present and the wave is uni-directional ($b_l = 0$ is a solution of equation (2.9)), then this reduces to the first term of the Stokes solution for a travelling wave. It is more usual to use the amplitude of the surface elevation rather than the velocity potential; thus corresponding to the primary mode

$$\phi_l = (A_l e^{i\mathbf{k}_l.\mathbf{r}} + A_{-l} e^{-i\mathbf{k}_l.\mathbf{r}})\,e^{|\mathbf{k}_l|y}, \tag{2.15}$$

the surface elevation is
$$\eta_l = B_l e^{i\mathbf{k}_l.\mathbf{r}} + B_l e^{-i\mathbf{k}_l.\mathbf{r}}, \tag{2.16}$$

where we have
$$A_l = a_l e^{-i\omega_l t} + b_l e^{i\omega_l t}, \tag{2.17}$$

$$B_l = d_l e^{-i\omega_l t} + e_l e^{i\omega_l t}. \tag{2.18}$$

It follows that
$$-i\omega_l d_l = |\mathbf{k}_l|\,a_l, \tag{2.19}$$

$$i\omega_l e_l = |\mathbf{k}_l|\,b_l, \tag{2.20}$$

and so
$$d_l = d_l(0)\,e^{-i\omega_l\lambda_l t}, \tag{2.21}$$

$$e_l = e_l(0)\,e^{i\omega_l\lambda_l t}, \tag{2.22}$$

where
$$\omega_l^2\lambda_l = \sum_m \alpha_{lm} c_m^2(d_m(0)\,d_m^*(0) + e_m(0)\,e_m^*(0)). \tag{2.23}$$

3. Modifications due to apparent resonances

It has been shown by Phillips (1960) and Longuet-Higgins (1962) that in the case of deep water no apparent resonances can occur in the second-order interactions; but they can occur in the third-order interactions. Thus a new mode not initially present may be excited through the non-linearity of existing modes. In their analysis such a newly excited mode grows like t. While this is true of the initial growth it is spurious in the sense that it gives no indication as to what extent this mechanism is effective, the magnitude to which the new mode will grow and the amplitude modifications of existing modes. It is this point we now investigate.

For simplicity we shall restrict ourselves to the case in which each $b_l = 0$. Thus the primary modes are each uni-directional travelling waves. If this is not the case (for example, if one mode is a standing wave) a similar analysis applies.

First suppose $\omega_1+\omega_2+\omega_3 = \omega_4$ and $\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3 = \mathbf{k}_4$, where $\mathbf{k}_i \neq \mathbf{k}_j$ for $i \neq j$. The equations corresponding to (2.8) are now

$$\omega_1 \frac{da_1}{dt} = ia_1\left(\sum_{m=1}^{4} \alpha_{1m} a_m a_m^*\right) + i\theta_1 a_2^* a_3^* a_4, \tag{3.1}$$

$$\omega_2 \frac{da_2}{dt} = ia_2\left(\sum_{m=1}^{4} \alpha_{2m} a_m a_m^*\right) + i\theta_2 a_1^* a_3^* a_4, \tag{3.2}$$

$$\omega_3 \frac{da_3}{dt} = ia_3\left(\sum_{m=1}^{4} \alpha_{3m} a_m a_m^*\right) + i\theta_3 a_1^* a_2^* a_4, \tag{3.3}$$

$$\omega_4 \frac{da_4}{dt} = ia_4\left(\sum_{m=1}^{4} \alpha_{4m} a_m a_m^*\right) + i\theta_4 a_1 a_2 a_3, \tag{3.4}$$

where the θ_m, $m = 1, 2, 3, 4$, are real constants which can be evaluated (by suitable sign changes) from equation (1.15). The set of equations (3.1)—(3.4) is an eighth-order system governing the slowly varying amplitudes and phases of the four modes concerned. Three simple integrals can be obtained which demonstrate the energy-sharing mechanism involved, namely

$$\frac{\omega_m}{\theta_m}|a_m|^2+\frac{\omega_4}{\theta_4}|a_4|^2 = \frac{\omega_m}{\theta_m}|a_m(0)|^2+\frac{\omega_4}{\theta_4}|a_4(0)|^2 \quad (m = 1, 2, 3). \tag{3.5}$$

The set of four modes form a group in which the growth of one must be compensated for by the decay of another, and the effect of the non-linearity can no longer be interpreted as merely changing the wave speeds of the various modes. Three interacting waves of the group can generate the fourth from rest to become an order-one mode, each oscillation having a slowly varying amplitude and phase.

We consider in more detail the case in which $2\omega_1 = \omega_2+\omega_3$ and $2\mathbf{k}_1 = \mathbf{k}_2+\mathbf{k}_3$; the three modes forming a resonant triad. The three equations governing the first-order motion are now

$$\omega_1 \frac{da_1}{dt} = ia_1\left(\sum_{m=1}^{3} \alpha_{1m} a_m a_m^*\right) + i\psi_1 a_1^* a_2 a_3, \tag{3.6}$$

$$\omega_2 \frac{da_2}{dt} = ia_2\left(\sum_{m=1}^{3} \alpha_{2m} a_m a_m^*\right) + i\psi_2 a_1^2 a_3^*, \tag{3.7}$$

$$\omega_3 \frac{da_3}{dt} = ia_3\left(\sum_{m=1}^{3} \alpha_{3m} a_m a_m^*\right) + i\psi_3 a_1^2 a_2^*, \tag{3.8}$$

and so

$$\frac{\omega_1}{\psi_1}|a_1|^2+\frac{\omega_m}{\psi_m}|a_m|^2 = \frac{\omega_1}{\psi_1}|a_1(0)|^2+\frac{\omega_m}{\psi_m}|a_m(0)|^2 \quad (m = 2, 3), \tag{3.9}$$

or equivalently in terms of the surface elevations

$$\frac{|d_1|^2}{\omega_1\psi_1}+\frac{|d_m|^2}{\omega_m\psi_m} = \frac{|d_1(0)|^2}{\omega_1\psi_1}+\frac{|d_m(0)|^2}{\omega_m\psi_m} \quad (m = 2, 3), \tag{3.10}$$

where

$$\begin{aligned}\psi_1 = {} & \tfrac{3}{2}|\mathbf{k}_1|^2(|\mathbf{k}_2|\,|\mathbf{k}_3|-\mathbf{k}_2.\mathbf{k}_3)+\frac{\mathbf{k}_1.\mathbf{k}_2}{2}(|\mathbf{k}_1|\,|\mathbf{k}_3|+\mathbf{k}_1.\mathbf{k}_3)+\frac{\mathbf{k}_1.\mathbf{k}_3}{2}(|\mathbf{k}_1|\,|\mathbf{k}_2|+\mathbf{k}_1.\mathbf{k}_2)\\
& +\frac{(|\mathbf{k}_1|+|\mathbf{k}_2|+|\mathbf{k}_3|)}{2}\bigg\{|\mathbf{k}_1|\,(|\mathbf{k}_2|\,|\mathbf{k}_3|-\mathbf{k}_2.\mathbf{k}_3)+\frac{\omega_2\omega_3}{g}|\mathbf{k}_1|\,(|\mathbf{k}_2|+|\mathbf{k}_3|)\\
& -\frac{\omega_1\omega_2}{g}(|\mathbf{k}_1|\,|\mathbf{k}_3|+2\mathbf{k}_1.\mathbf{k}_3)-\frac{\omega_1\omega_3}{g}(|\mathbf{k}_1|\,|\mathbf{k}_2|+2\mathbf{k}_1.\mathbf{k}_2)\bigg\}\\
& +\frac{(\omega_1-\omega_3)(|\mathbf{k}_1|\,|\mathbf{k}_3|+\mathbf{k}_1.\mathbf{k}_3)}{g|\mathbf{k}_1-\mathbf{k}_3|-(\omega_1-\omega_3)^2}\bigg\{\omega_1(-2\mathbf{k}_1.\mathbf{k}_2)\\
& +\omega_2(2\mathbf{k}_1.\mathbf{k}_2+|\mathbf{k}_1|\,|\mathbf{k}_2|+|\mathbf{k}_1|\,|\mathbf{k}_1-\mathbf{k}_2|-|\mathbf{k}_1|^2)\bigg\}+\frac{(\omega_1-\omega_2)(|\mathbf{k}_1|\,|\mathbf{k}_2|+\mathbf{k}_1.\mathbf{k}_2)}{g|\mathbf{k}_1-\mathbf{k}_2|-(\omega_1-\omega_2)^2}\\
& \times\{\omega_1(-2\mathbf{k}_1.\mathbf{k}_3)+\omega_3(2\mathbf{k}_1.\mathbf{k}_3+|\mathbf{k}_1|\,|\mathbf{k}_3|+|\mathbf{k}_1|\,|\mathbf{k}_1-\mathbf{k}_3|-|\mathbf{k}_1|^2)\},\end{aligned} \tag{3.11}$$

$$\begin{aligned}\psi_2 = {} & (|\mathbf{k}_1|\,|\mathbf{k}_3|+\mathbf{k}_1.\mathbf{k}_3)\bigg[\frac{\omega_1\omega_3}{g}(2|\mathbf{k}_1|+|\mathbf{k}_3|)-2|\mathbf{k}_1|^2-\tfrac{3}{2}|\mathbf{k}_1|\,|\mathbf{k}_3|-\tfrac{1}{2}\mathbf{k}_1.\mathbf{k}_3\\
& +\frac{(\omega_1-\omega_3)}{g|\mathbf{k}_1-\mathbf{k}_3|-(\omega_1-\omega_3)^2}\{\omega_3(4|\mathbf{k}_1|\,|\mathbf{k}_1-\mathbf{k}_3|+2\mathbf{k}_1.\mathbf{k}_3)+\omega_1(4|\mathbf{k}_1|^2+|\mathbf{k}_3|^2-6\mathbf{k}_1.\mathbf{k}_3\\
& -|\mathbf{k}_1|\,|\mathbf{k}_3|-4|\mathbf{k}_1|\,|\mathbf{k}_1-\mathbf{k}_3|-|\mathbf{k}_3|\,|\mathbf{k}_1-\mathbf{k}_3|)\}\bigg],\end{aligned} \tag{3.12}$$

and where ψ_3 is derivable from ψ_2 by interchanging the subscripts two and three.

μ	$\mathbf{k}_1$	$\mathbf{k}_2$	$\mathbf{k}_3$	$-g^{-\frac{1}{2}}\omega_1\psi_1$	$-g^{-\frac{1}{2}}\omega_2\psi_2$	$-g^{-\frac{1}{2}}\omega_3\psi_3$
$\frac{1}{4}\pi$	(1, 0)	(0·347, 0·347)	(1·653, −0·347)	1·251	0·460	0·813
$\frac{1}{2}\pi$	(1, 0)	(0, 0·332)	(2, −0·332)	0·238	0·068	0·169
$\frac{3}{4}\pi$	(1, 0)	(−0·189, 0·189)	(2·189, −0·189)	0·015	0·001	0·011

TABLE 1

It is evident from equation (3.10) that in $(|d_1|^2, |d_2|^2, |d_3|^2)$-space the modes lie on a straight line having direction numbers $(\omega_1\psi_1, -\omega_2\psi_2, -\omega_3\psi_3)$ through the point corresponding to the initial amplitudes. There is an asymmetry in this situation in that modes one and two (or three) can generate mode three (or two) from rest; but the interactions of modes two and three will not generate mode one from rest. If all three modes are initially present then the energy distribution will vary with time. No stable equilibrium amplitudes can be expected as the system is conservative. In practice the inevitable dissipation could modify this conclusion.

To illustrate the extent of the energy exchange, numerical results have been obtained using equations (3.11) and (3.12), for three special cases; namely, when the modes one and two are inclined at angles of $\mu = \frac{1}{4}\pi, \frac{1}{2}\pi, \frac{3}{4}\pi$. It is seen from table 1 that the interaction can be significant and should be observable.

4. Concluding remarks

It has been shown that the principal effect of ordinary weakly non-linear interactions of gravity waves is a modification of the linear wave speeds of the individual modes dependent on all the modes present. A similar result was obtained by Longuet-Higgins & Phillips (1962) by allowing secular terms to arise in the perturbation scheme and interpreting them as changes in wave speeds; however, there appears to be some algebraic differences between their results and the present work.

If resonant conditions are satisfied, or indeed close to being satisfied, there is a direct but slow exchange of energy between the order-one modes. New primary waves can be generated by the non-linear coupling of the equations, and the existing energy distribution modified. In such circumstances the free surface at a particular point must be expected to have a slowly changing pattern. Higher-order resonances are of course possible; but the essential ideas would be similar to those considered here.

It should be remarked that the concept of resonant interactions has been suggested as a possible explanation of the transition phenomenon (Raetz 1959). In this case the mean flow is an energy source for the various modes; but it appears that this situation is governed by a far more dramatic mechanism (Benney & Greenspan 1962). Nevertheless, in many physical systems the transfer of energy associated with apparent resonant interactions may be an important factor.

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